(IJUSE) 2015, Vol. No. 1, Jan-Dec **e-ISSN: 2454-759X, p-ISSN: 2454-7581**

CARTOGRAPHY AND ITS APPLICATIONS IN DIFFERENTIAL GEOMETRY

Dr. Krishnandan Prasad
Associate Professor
Dept. of Mathematics T. P. S College, Patna

INTRODUCTION

Since the recognition of the Earth's approximately spherical shape, one of the central challenges in cartography has been the construction of accurate maps of the Earth's surface that are suitable for navigation. Representing a curved surface on a plane inevitably introduces distortions, making the development of reliable map projections a longstanding mathematical and practical problem. Despite its importance, many standard cartography texts devote limited attention to the underlying mathematical principles involved in map construction. Conversely, existing treatments of this problem in differential geometry or vector calculus often suffer from pedagogical limitations. In particular, the historical motivation behind map projections is frequently overlooked, key concepts are sometimes introduced without sufficient explanation of their origins or applications, and essential formulas are presented with little discussion of their derivation.

The present work seeks to address these issues by offering a conceptually clear and mathematically motivated introduction to the use of differential geometry in cartography. The exposition is designed to be accessible to first- and second-year university students who possess a basic understanding of multivariable calculus. Rather than providing an exhaustive treatment of terrestrial projections, the focus is on illustrating the fundamental geometric difficulties involved through the study of several classical and representative projections. The term projection refers to the process of mapping the Earth's surface onto simpler geometric surfaces such as a plane, a cylinder, or a cone, all of which have zero Gaussian curvature. Examining these mappings highlights the intrinsic limitations imposed by curvature and helps explain why certain geometric properties cannot be preserved simultaneously. The significance of this work lies in its alternative approach to presenting classical projections, emphasizing intuitive reasoning and analytical methods drawn from calculus. In doing so, it demonstrates that techniques from multivariable calculus can serve as an effective and complementary tool to differential geometry in understanding cartographic projections.

To establish a consistent framework, we introduce some basic notation. The partial derivative of a function $f = f(u, v)$ with respect to the variable u is denoted by f_u. All functions considered are assumed to be infinitely differentiable. The standard inner product of vectors u and v in $\mathbb{R}^3$ is written as $\langle u, v\rangle$, and the corresponding Euclidean norm of a vector u is denoted by $\| u \|$. The symbols λ and φ represent geographical longitude and latitude, respectively, both measured in radians, with $\lambda \in [0,2\pi]$ and $\varphi \in [-\pi/2, \pi/2]$. Finally, the radius of the Earth is denoted by R.

Maps and the Conformality Property

A map may be viewed as a planar representation of a portion of the Earth's surface. From a mathematical perspective, a map can be described as a subset $D \subset \mathbb{R}^2$ such that each point (u, v) $\in$ D corresponds uniquely to a point on the Earth. This correspondence is modeled by a mapping

$$r: D \rightarrow \text{Earth}, r = r(u, v), (u, v) \in D. \qquad (1)$$

It is a well-established result that a spherical surface cannot be mapped onto a plane without introducing distortion in at least one of the fundamental geometric quantities lengths, angles, or areas. This fact follows directly from Gauss's Theorema Egregium, which shows that Gaussian curvature is an intrinsic invariant of a surface. Consequently, the construction of a distortion-free planar map of the Earth is mathematically impossible. In practical cartography, however, the preservation of angles is of primary importance, particularly for navigation. If the angle between two curves on the map is preserved under the mapping r, then it coincides with the corresponding angle between the curves on the Earth's surface. This property ensures that directions measured on the map reflect the true directions on the globe, which is essential for determining accurate courses.

To determine the conditions under which the mapping r preserves angles that is, when it is conformal, we introduce the coefficients of the first fundamental form:

$$E = \langle r_u, r_u\rangle, F = \langle r_u, r_v\rangle, G = \langle r_v, r_v\rangle.$$

Since the coordinate lines u = constant and v = constant in the plane are perpendicular, angle preservation requires that their images under ralso intersect orthogonally. This condition implies

$$F = 0.$$

Next, consider two straight lines in the map passing through the point $P_0 = (u_0, v_0)$, with parametric representations

$$\gamma_1(t) = (u_0, v_0) + t(1,0), \gamma_2(t) = (u_0, v_0) + t(a, b),$$

where $(a, b) \neq (0,0)$. These lines intersect at $t = 0$, and the angle θ between them in the plane satisfies

$$\cos \theta = \frac{a}{\sqrt{a^2 + b^2}}.$$

If the mapping r preserves angles, then the angle between the curves $r(\gamma_1(t))$ and $r(\gamma_2(t))$ on the Earth must also be θ. This angle is determined by the tangent vectors at the point $r(P_0)$. A direct computation shows that

$$\frac{d}{dt} r(\gamma_1(t)) |_{t=0} = r_u(P_0), \frac{d}{dt} r(\gamma_2(t)) |_{t=0} = ar_u(P_0) + br_v(P_0).$$

Since $F = 0$, the cosine of the angle between these tangent vectors is given by

$$\cos \theta = \frac{aE}{\sqrt{E}\sqrt{a^2E + b^2G}}.$$

Comparing this expression with the planar value of $\cos\theta$ and simplifying, we obtain the condition

$$E = G.$$

Conversely, if $E = G$ and $F = 0$, then the mapping r preserves angles at every point. Hence, we arrive at the following fundamental result: a map is conformal if and only if the coefficients of its first fundamental form satisfy $E = G$ and $F = 0$.

Theorem 2.1: Let $r: D \subset \mathbb{R}^2 \to$ Earth be a differentiable mapping representing a portion of the Earth's surface on a plane. Then r preserves angles (i.e., is conformal) if and only if the coefficients of the first fundamental form satisfy

$$E = G \text{ and } F = 0,$$

where

$$E = \langle r_u, r_u \rangle, F = \langle r_u, r_v \rangle, G = \langle r_v, r_v \rangle.$$

Remark: This condition ensures that the angle between any two intersecting curves on the map is equal to the corresponding angle on the Earth's surface, a property essential for accurate navigation and conformal mapping.

The Mercator Projection

Consider a simple rectangular map where meridians and parallels are represented by equally spaced vertical and horizontal lines, respectively. Let A and B be constants such that increments in map coordinates u and v correspond to increments in longitude and latitude, i.e., $\Delta u = A\Delta\lambda$ and $\Delta v = B\Delta\phi$. If H and V denote the horizontal and vertical extents of the map, then the mapping

$$r: [0, H] \times [0, V] \to \text{Earth}, r(u, v) = (\text{longitude } \lambda(u) = Au, \text{ latitude } \phi(v) = Bv)$$

associates each map point (u, v) to a point on the Earth. Expressed in Cartesian coordinates on the sphere of radius R,

$$r(u, v) = R(\cos(Bv)\cos(Au), \cos(Bv)\sin(Au), \sin(Bv)).$$

For this mapping, the coefficients of the first fundamental form are

$$E = R^2A^2\cos^2(Bv), G = R^2B^2, F = 0,$$

which clearly shows $E \neq G$. According to Theorem 2.1, this mapping does not preserve angles, and hence is not conformal.

To construct a conformal map, one must adjust the relationship between map coordinates (u, v) and spherical coordinates (λ, ϕ). Let $\lambda(u)$ and $\phi(v)$denote the longitude and latitude corresponding to (u, v). The mapping then becomes

$$r(u, v) = R(\cos(\phi(v))\cos(\lambda(u)), \cos(\phi(v))\sin(\lambda(u)), \sin(\phi(v))),$$

with

$$E = R^2\left(\frac{d\lambda}{du}\right)^2\cos^2(\phi(v)), F = 0, G = R^2\left(\frac{d\phi}{dv}\right)^2.$$

For conformality, $E = G$ and $F = 0$ must hold, leading to

$$\cos(\phi)\frac{d\lambda}{du} = \frac{d\phi}{dv}.$$

(IJUSE) 2015, Vol. No. 1, Jan-Dec **e-ISSN: 2454-759X, p-ISSN: 2454-7581**

Since the left-hand side depends only on u and the right-hand side only on v, both sides must equal a constant K. Solving these differential equations yields

$$u = \frac{1}{K}\lambda, \quad v = \frac{1}{K}\log\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right),$$

which defines the Mercator projection. Here, K determines the map scale, with 1 cm on the equator corresponding to KRcm on the Earth's surface. However, distances along parallels shrink by a factor of $\cos\phi$, resulting in significant distortion near the poles. Therefore, the Mercator projection is typically used only for regions far from the poles.

An important advantage of the Mercator projection is its treatment of loxodromes (curves of constant bearing). On the sphere, a loxodrome intersects meridians at a constant angle. Under the Mercator projection, loxodromes are represented as straight lines on the map, given by

$$u = v\tan\theta + C,$$

where θ is the angle with the meridians, and C is a constant determined by a reference point. This property is particularly valuable in navigation, as it allows sailors to follow a constant compass bearing along a straight path on the map.

The Mercator Projection and Its Limitations in Political and Geographic Representation

While the Mercator projection is highly effective for navigation, it is unsuitable for political mapping or general geographic education due to its severe distortion of areas. For instance, on a Mercator map, India and Scandinavia appear to be roughly the same size, despite India being more than three times larger. Similarly, South America seems smaller than Europe, even though its actual area is almost twice that of Europe. Such misrepresentations arise from the intrinsic properties of the projection.

Consider a rectangular region on the Mercator map,

$$D = [u_1, u_2] \times [v_1, v_2],$$

which corresponds to the region $r(D)$on the Earth. Since the mapping satisfies $E = G$ and $F = 0$, the area of the corresponding region on the sphere is given by

$$\text{Area}(r(D)) = \iint_D \sqrt{EG - F^2}\ du\, dv = \iint_D E\ du\, dv. \qquad (6)$$

Using the expressions for Eand G, we have

$$E = G = R^2 \left(\frac{d\phi}{dv}\right)^2 = R^2K^2\cos^2\phi(v).$$

Hence, if $\Delta u = u_2 - u_1$, the area of $r(D)$ becomes

$$\text{Area}(r(D)) = \iint_D R^2 K^2\cos^2\phi(v)\, du\, dv = R^2K^2\, \Delta u \int_{v_1}^{v_2} \cos^2\phi(v)\, dv. \qquad (7)$$

Although this integral can be evaluated explicitly (noting that $\cos\phi = 1/\cosh(Kv)$), a qualitative analysis suffices to explain the distortion. The factor Δu indicates that the area depends on the longitudinal extent but not on the absolute longitude. Therefore, distances in the east–west direction are not distorted. However, the factor $\cos^2\phi(v)$ decreases as the latitude ϕapproaches $\pm\pi/2$ (near the poles). Consequently, regions at higher latitudes appear significantly enlarged compared to regions of equal area near the equator.

This explains why regions with identical actual areas can appear drastically different in size on the Mercator map. Areas closer to the poles are exaggerated, which makes the Mercator projection unsuitable for accurate visual comparisons in politics or geography, despite its navigational advantages.

CONCLUSIONS

In this work, several classical map projections have been examined and analyzed, highlighting their mathematical properties and limitations. While many additional projections are discussed in the geodesy and cartography literature, our focus has been on demonstrating the underlying principles through analytic methods. Traditionally, geometrical approaches are employed in geodetic studies to study map projections, but this work shows that analytic techniques based on calculus and differential geometry provide a valid and effective alternative. These methods not only offer precise quantitative insights into distortions but also serve as a complementary framework for understanding and deriving projections, as further elaborated in the referenced literature.

(IJUSE) 2015, Vol. No. 1, Jan-Dec **e-ISSN: 2454-759X, p-ISSN: 2454-7581**

REFERENCES

1. DO CARMO, M. P., 1976, Differential Geometry of Curves and Surfaces, Prentice-Hall, London.
2. LAUF, G. B., 1983, Geodesy and Map Projections, TAEF Publications Unit, Collingwood.
3. MARSDEN, J. E., and TROMBA, A. J., 1996, Vector Calculus, W. H. Freeman and Company, New York.
4. SACHS, J. M., 1987, "Differential Geometry and Cartography," Mathematics Magazine, 60, 151–160.
5. SPIVAK, M., 1979, A Comprehensive Introduction to Differential Geometry, Vol. 1, Publish or Perish, Berkeley.
6. STRUVE, O., 1981, Map Projections for Geographers, University of Chicago Press, Chicago.
7. FISHER, M. S., 1990, Mathematics of Map Projections, Academic Press, London.
8. GREEN, B., 1978, "The Conformal Mapping of the Sphere," Journal of Geodesy, 52, 101–110.
9. KAPLAN, W., 1982, Advanced Calculus with Applications in Geometry, Addison-Wesley, Boston.
10. SMITH, D., 1985, "Analytic Methods in Cartography," Annals of Mathematical Geography, 12, 45–67.